Algebraic solution of a graphene layer in a transverse electric and perpendicular magnetic fields

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Abstract. We present an exact algebraic solution of a single graphene plane in transverse electric and perpendicular magnetic fields. The method presented gives both the eigen-values and the eigen-functions of the graphene plane. It is shown that the eigen-states of the problem can be casted in terms of coherent states, which appears in a natural way from the formalism.

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1. Introduction

A major breakthrough in condensed matter physics took place when K. S. Novoselov *et al.* [1] , at Manchester University, UK, discovered an electric field effect in atomically thin carbon films. This electric field effect is characterized by the control of the electronic density in the films using a backgate setup. These atomically thin carbon films were thought not to exist since long range order in two dimension can not occur. The system solves the apparent paradox by forming ripples. A single atomic layer of these thin carbon films is called graphene and its electric and magneto-electric properties triggered a new research field in condensed matter physics. The manufacture of graphene was followed by the production of other 2D crystals[2], which, however, have not the same exciting properties as graphene does. Applying high magnetic fields to a graphene sample, the Manchester group discovered that the quantization rule for the Hall conductivity is not the same one observes in the two-dimensional electron gas, being given instead by[3]

$$\sigma_{\rm Hall} = 4\left(n+\frac{1}{2}\right)\frac{e^2}{h}\,, \tag{1}$$

with n an integer including zero. A confirmation of this result was independently obtained by Philip Kim's group[4], at Colombia University, New York, USA. This new

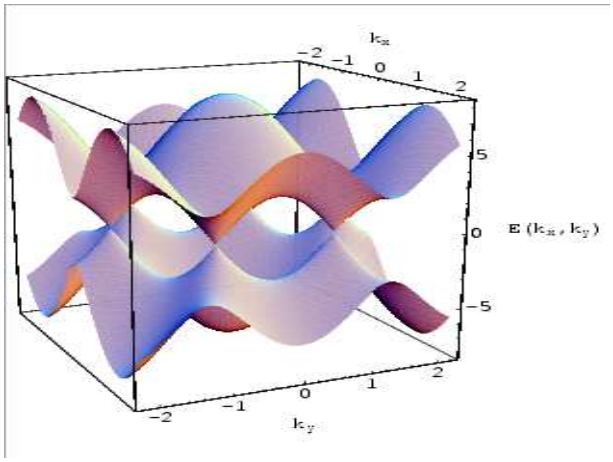

Figure 1. Valence and conduction bands of electrons in graphene. The two bands touch each other in six points of the Brillouin zone, called Dirac points.

quantum Hall effect was predicted by two groups working independently and using different methods [5, 6]. As explained by the two groups the new quantization rule for the Hall conductivity is a consequence of the dispersion relation of the electrons in the honeycomb lattice. This dispersion resembles the spectrum of ultra relativistic particles, i.e., the dispersion is that of particles of zero rest mass and an effective velocity of light. For graphene the effective velocity of light is $v_F = c/300$, with c the true velocity of light.

For a qualitative description of the physics of graphene, both theoretical and experimental, see references [7] by Castro Neto *et al.*, [8] by M. I. Katsnelson, and [9] by Geim and Novoselov.

In Figure 1 we show the energy dispersion of electrons in the honeycomb lattice. The spectrum shows a valence (lower) and a conduction (upper) bands. Since graphene has one electron per unit cell the valence band is completely filled and the properties of the system are determined by the nature of its spectrum close to the points where the valence and the conduction bands touch each other. These points are called Dirac points and are in number of six. In Figure 2 we show the spectrum close to the Dirac points. It is clear that the spectrum has conical shape of the form

$$E = \pm v_F p, \tag{2}$$

where p is the magnitude of the momentum $\boldsymbol{p}$ around the Dirac point.

A relativistic particle has an energy given by

$$E = \sqrt{m^2c^4 + p^2c^2}\,, \tag{3}$$

and therefore an ultra-relativistic particle ($m \to 0$) has a spectrum given by

$$E = cp\,. \tag{4}$$

It is clear from equation (4) that electrons in graphene, close to the Dirac points, have an energy dispersion with a formal equivalence to ultra-relativistic particles. As a

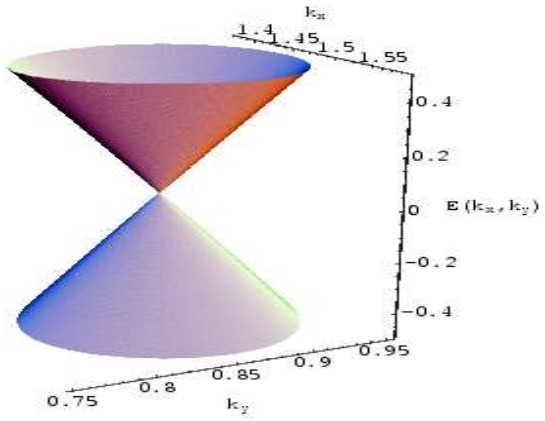

Figure 2. Valence and conduction bands close to one of the six Dirac points. It is clear that the bands can be approximated by a conical dispersion.

consequence the quantum properties of the system has to be described by the massless (ultra-relativistic) Dirac equation in two plus one dimensions.

We are interested in studying the spectrum of massless Dirac particles in the presence of a magnetic field perpendicular to the plane and an in plane homogeneous electric field, both static, a situation that occurs in the Hall effect. In the next section we present a full algebraic solution to this quantum problem.

2. Algebraic solution

2.1. Hamiltonian

The massless Dirac equation in two plus one dimensions has the form

$$v_F(\sigma_x p_x + \sigma_y p_y)\Psi(\boldsymbol{r},t) = i\hbar\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t}\,, \tag{5}$$

where σ_i, $i = x, y, z$, represents the Pauli spin matrices and p_i, $i = x, y$, is the i component of the momentum operator in the position basis $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$. Since we are looking for stationary states we make the substitution $\Psi(\boldsymbol{r},t) = \psi(\boldsymbol{r})e^{-i\epsilon t/\hbar}$. This substitution transforms the Dirac equation into an eigenvalue problem of the form

$$v_F(\sigma_x p_x + \sigma_y p_y)\psi(\boldsymbol{r}) = \epsilon\psi(\boldsymbol{r})\,. \tag{6}$$

The introduction of a magnetic field into a quantum mechanical problem is made by transforming the momentum operator according to the rule (minimal coupling)

$$\boldsymbol{p} \to \boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{r}), \tag{7}$$

where $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential and q is the charge of the particle. The magnetic field $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ using the relation $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. There is a lot of freedom in choosing $\boldsymbol{A}$ and a common choice, known as Landau gauge, is $\boldsymbol{A} = (-By, 0, 0)$. Let us now assume that in addition to the magnetic field one has a homogeneous electric field,

perpendicular to the magnetic field, and oriented along the y direction. This adds to the Hamiltonian a term of the form

$$qV\mathbf{1} = q\mathcal{E}y\mathbf{1}\,, \tag{8}$$

where V is the electric potential associated with the applied electric field $\boldsymbol{E} = (0, \mathcal{E}, 0)$ and $\mathbf{1}$ is the 2×2 unit matrix.

Putting all together, the problem of a graphene layer in the presence of a magnetic field perpendicular to the layer and of an electric field parallel to the layer has its Hamiltonian, in the position basis, given by:

$$H(x,y) = v_F \begin{pmatrix} q\mathcal{E}y/v_F & p_x - ip_y + qBy \\ p_x + ip_y + qBy & q\mathcal{E}y/v_F \end{pmatrix}. \tag{9}$$

The eigenproblem $H(\boldsymbol{r})\psi(\boldsymbol{r}) = \epsilon\psi(\boldsymbol{r})$ can be further simplified by representing the eigenfunction $\psi(\boldsymbol{r})$ as

$$\psi(\boldsymbol{r}) = e^{ikx}\phi(y)\,, \tag{10}$$

suggested by the translational invariance of Eq. (9) along the x direction. Because we are dealing with electrons one has $q = -e$, with $e > 0$. Using equation (10) in equation (9) we obtain

$$v_F \begin{pmatrix} -e\mathcal{E}y/v_F & k\hbar - ip_y - eBy \\ k\hbar + ip_y - eBy & -e\mathcal{E}y/v_F \end{pmatrix} \phi(y) = \epsilon\phi(y)\,. \tag{11}$$

Next we perform a change of variables $y = \bar{y}l_B + l_B^2 k$ and $\partial/\partial\bar{y} = l_B\partial/\partial y$ (corresponding to the introduction of the adimensional length $\bar{y}$), with $l_B = \sqrt{\hbar/(eB)}$ the magnetic length, and introduce the operators

$$a \;= \frac{1}{\sqrt{2}}(\hat{\bar{y}} + \partial/\partial\hat{\bar{y}})\,, \tag{12}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(\hat{\bar{y}} - \partial/\partial\hat{\bar{y}})\,, \tag{13}$$

which satisfy the standard commutation relation $[a, a^\dagger] = 1$. Note that in Eqs. (12) and (13) we have used the hat to distinguish between operators and their matrix elements in a given basis.

Performing standard manipulations the Hamiltonian operator can be brought into a more transparent form

$$\hat{H} = -\begin{pmatrix} e\mathcal{E}l_B^2 k + E_B(a + a^\dagger) & E_F a \\ E_F a^\dagger & e\mathcal{E}l_B^2 k + E_B(a + a^\dagger) \end{pmatrix}, \tag{14}$$

with $E_F = \sqrt{2}v_F\hbar/l_B$ and $E_B = e\mathcal{E}l_B/\sqrt{2}$. The eigenvalue equation one needs to solve has the form

$$\begin{pmatrix} E_B(a + a^\dagger) & E_F a \\ E_F a^\dagger & E_B(a + a^\dagger) \end{pmatrix} \begin{pmatrix} |a_1\rangle \\ |a_2\rangle \end{pmatrix} = \epsilon_0 \begin{pmatrix} |a_1\rangle \\ |a_2\rangle \end{pmatrix}, \tag{15}$$

where $\epsilon_0 = -(\epsilon + e\mathcal{E}l_B^2 k)$. The eigenproblem is now in its most simplified form, with an effective Hamiltonian operator given by,

$$\hat{\mathcal{H}} = \begin{pmatrix} E_B(a + a^\dagger) & E_F a \\ E_F a^\dagger & E_B(a + a^\dagger) \end{pmatrix} \tag{16}$$

2.2. Diagonalization method

Before going further with the diagonalization it is worth mentioning some properties of the Hamiltonian operator $\hat{\mathcal{H}}$, defined in Eq. (16). First we define $\hat{\tilde{\mathcal{H}}}$ as,

$$\hat{\tilde{\mathcal{H}}} = \sigma_z \hat{\mathcal{H}} \sigma_z = \begin{pmatrix} E_B(a + a^\dagger) & -E_F a \\ -E_F a^\dagger & E_B(a + a^\dagger) \end{pmatrix}, \tag{17}$$

which, by definition, is an operator acting on the same Hilbert space as $\hat{\mathcal{H}}$. Then we make the observation that both $\hat{\mathcal{H}} + \hat{\tilde{\mathcal{H}}}$ and $\hat{\tilde{\mathcal{H}}}\hat{\mathcal{H}}$ can be written as the 2×2 unit matrix times a simple operator (not a matrix operator) plus a 2×2 real matrix. The same holds for their linear combination, which enables us to write

$$\mu(\hat{\mathcal{H}} + \hat{\tilde{\mathcal{H}}}) + \nu \hat{\tilde{\mathcal{H}}}\hat{\mathcal{H}} = \hat{J}\mathbf{1} + \boldsymbol{K}, \tag{18}$$

for some simple operator $\hat{J}$, some 2×2 real matrix $\boldsymbol{K}$, and real μ and ν. Now let the spinor $|\psi\rangle$ be an eigenstate of $\hat{\mathcal{H}}$. Applying the left hand member of Eq. (18) to $|\psi\rangle$ we obtain,

$$[\mu(\hat{\mathcal{H}} + \hat{\tilde{\mathcal{H}}}) + \nu \hat{\tilde{\mathcal{H}}}\hat{\mathcal{H}}]|\psi\rangle = (\mu\epsilon_0 + \mu\hat{\tilde{\mathcal{H}}} + \nu\epsilon_0 \hat{\tilde{\mathcal{H}}})|\psi\rangle, \tag{19}$$

which means that if we chose $\mu = \epsilon_0$ and $\nu = -1$ we reduce our problem to,

$$(\hat{J}\mathbf{1} + \boldsymbol{K})|\psi\rangle = \epsilon_0^2 |\psi\rangle, \tag{20}$$

with

$$\hat{J} = (E_F^2 - 2E_B^2)\hat{n} - E_B^2(aa + a^\dagger a^\dagger) + 2\epsilon_0 E_B(a + a^\dagger), \tag{21}$$

where $\hat{n} = a^\dagger a$ is the number operator, and

$$\boldsymbol{K} = \begin{pmatrix} E_F^2 - E_B^2 & E_F E_B \\ -E_F E_B & -E_B^2 \end{pmatrix}. \tag{22}$$

Equation (20) is indeed simpler than our starting point, Eq. (15). We readily see that the spinor $|\psi\rangle$, given by

$$|\psi\rangle = \begin{pmatrix} |a_1\rangle \\ |a_2\rangle \end{pmatrix}, \tag{23}$$

can be written as

$$|\psi\rangle = |\phi\rangle \begin{pmatrix} u \\ v \end{pmatrix}, \tag{24}$$

where $|\phi\rangle$ is the eigenvector of the operator $\hat{J}$ and the spinor

$$\chi^R = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{25}$$

is the right eigenvector of the eigenvalue problem

$$\boldsymbol{K}\chi^R = \lambda\chi^R. \tag{26}$$

Nevertheless, there is one subtlety we must consider. Our simpler eigenproblem defined by Eq. (20) is such that their eigenvalues are ϵ_0^2, the squared eigenvalues of the original

problem given by Eq. (15). If ϵ_0 in Eq. (15) and ϵ_0^2 in Eq. (20) have the same degeneracy it is guaranteed that eigenvectors of both problems are the same, and we can carry on the diagonalization with any of those equations. If, however, both $+\epsilon_0$ and $-\epsilon_0$ are eigenvalues of $\hat{\mathcal{H}}$ in Eq. (15), then our simple eigenproblem in Eq. (20) will show an extra double degeneracy. This extra degeneracy must be handled carefully. Due to mixing the corresponding degenerate eigenvectors need not to be eigenstates of $\hat{\mathcal{H}}$, and these have to be found as particular linear combinations of the degenerated eigenvectors. It is instructive to switch off the electric field, for which Eq. (15) can be easily solved[5], and check whether the above problem shows up in the present case.

2.3. Zero electric field case

In the absence of the electric field we have $\mathcal{E} = E_B = 0$. As a consequence the operator $\hat{J}$ in Eq. (21) is already in its diagonal form, being analogous to the 1D harmonic oscillator: $\hat{J}|n\rangle = E_F^2 n|n\rangle$. In what regards the eigenproblem defined by Eq. (26) for $\mathcal{E} = 0$, we can obtain the corresponding eigenvectors as

$$\chi_+^R = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_-^R = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{27}$$

with eigenvalues $\lambda_+ = E_F^2$ and $\lambda_- = 0$, respectively. Recalling Eq. (20) we get $\epsilon_0^2(n,\pm) = E_F^2 n + \lambda_\pm$, where we recognize immediately the double degeneracy $\epsilon_0^2(n,+) = \epsilon_0^2(n+1,-) = E_F^2(n+1)$. This degeneracy is in fact due to the presence of both $+\epsilon_0$ and $-\epsilon_0$ as eigenvalues of $\hat{\mathcal{H}}$ in the absence of electric field. Solving Eq. (15) directly for $\mathcal{E} = 0$ gives[5] $\epsilon_0 = \pm E_F\sqrt{n+1}$ in addition to the zero eigenvalue $\epsilon_0 = 0$, with eigenvectors

$$|\psi\rangle_\pm = \begin{pmatrix} |n\rangle \\ \pm|n+1\rangle \end{pmatrix} \quad \text{and} \quad |\psi\rangle_0 = \begin{pmatrix} 0 \\ |0\rangle \end{pmatrix}, \tag{28}$$

respectively for nonzero and zero eigenvalues. Therefore we see that our method gives correctly $|\psi\rangle_0$, the only non-degenerate eigenvector,

$$|\psi\rangle_0 = |0\rangle\chi_-^R, \tag{29}$$

while $|\psi\rangle_\pm$ is given as the bonding and anti-bonding combination of degenerate eigenvectors $|n\rangle\chi_+^R$ and $|n+1\rangle\chi_-^R$:

$$|\psi\rangle_\pm = |n\rangle\chi_+^R \pm |n+1\rangle\chi_-^R. \tag{30}$$

As extra degeneracies due to the presence of a finite electric field are not to be expected, we will be able to identify any double degeneracy arising from Eq. (20) as a consequence of the presence of symmetrical eigenvalues $\pm\epsilon_0$ in the original problem.

2.4. Finite electric and magnetic fields

With the above analysis in mind we proceed with the diagonalization of our problem in a finite electric and magnetic fields using Eq. (20). Let us start by solving Eq. (26).

The corresponding eigenvalues are given by,

$$\lambda_{\pm} = -E_B^2 + \frac{1}{2}\left(E_F^2 \pm E_F\sqrt{E_F^2 - 4E_B^2}\right) , \tag{31}$$

and as right eigenvectors we obtain,

$$\chi_{\pm}^R = \sqrt{\left|\frac{E_B}{E_F}\right|}\begin{pmatrix} -\sqrt{C_{\pm}} \\ 1/\sqrt{C_{\pm}} \end{pmatrix} , \tag{32}$$

with

$$C_{\pm} = \frac{1}{2}\left(E_F/|E_B| \pm \sqrt{E_F^2/E_B^2 - 4}\right) . \tag{33}$$

From equation (31) we see that E_F and E_B must satisfy the relation,

$$E_F^2 \geq 4E_B^2 , \tag{34}$$

if real eigenvalues are to be obtained. The meaning of this inequality is discussed later. Having solved the eigenproblem (26), the eigenproblem

$$\hat{J}|\phi\rangle = (\epsilon_0^2 - \lambda_{\pm})|\phi\rangle \tag{35}$$

remains to be solved. The solution of the eigenproblem (35) is obtained in three steps. First we write the operator $\hat{J}$ as the sum of two terms, $\hat{H}_1 + \hat{H}_2$, given by

$$\hat{H}_1 = (E_F^2 - 2E_B^2)\hat{n} - E_B^2(aa + a^\dagger a^\dagger) , \tag{36}$$

and

$$\hat{H}_2 = 2\epsilon_0 E_B(a + a^\dagger) . \tag{37}$$

As a second step we diagonalize the Hamiltonian $\hat{H}_1$ using the canonical transformation

$$a^\dagger = \cosh U\gamma^\dagger - \sinh U\gamma , \tag{38}$$

and the corresponding Hermitian conjugated form for a. Replacing the $a^\dagger$ and the a operators in (36) one obtains

$$\begin{aligned}\hat{H}_1 = {} & E_B^2 2\sinh U\cosh U + (E_F^2 - 2E_B^2)\sinh^2 U + (\gamma^\dagger\gamma^\dagger + \gamma\gamma)[-E_B^2(\cosh^2 U + \sinh^2 U) \\ & - (E_F^2 - 2E_B^2)\sinh U\cosh U] + \gamma^\dagger\gamma[(E_F^2 - 2E_B^2)(\cosh^2 U + \sinh^2 U) \\ & + 4E_B^2\sinh U\cosh U] .\end{aligned} \tag{39}$$

In order for $\hat{H}_1$ to be diagonal we require that the coefficient multiplying the $(\gamma^\dagger\gamma^\dagger + \gamma\gamma)$ term should be null, leading to

$$[-E_B^2(\cosh^2 U + \sinh^2 U) - (E_F^2 - 2E_B^2)\sinh U\cosh U] = 0 \tag{40}$$

which can be cast in the form

$$\tanh(2U) = -\frac{2E_B^2}{E_F^2 - 2E_B^2} . \tag{41}$$

We note that since $\cosh U > 0$ for any value of U one must have $\sinh U < 0$ in order to satisfy equation (41). The result (41) together with $\cosh^2 U - \sinh^2 U = 1$ can be recasted in the form

$$\sinh^2 U = -\frac{1}{2}\left[1 - (E_F^2 - 2E_B^2)/\omega\right] , \tag{42}$$

and

$$\cosh^2 U = \frac{1}{2}\left[1 + (E_F^2 - 2E_B^2)/\omega\right] , \tag{43}$$

leading to

$$\sinh U \cosh U = -\frac{E_B^2}{\omega} \tag{44}$$

with $\omega = \sqrt{E_F^4 - 4E_F^2 E_B^2}$ and $E_F^2 > 4E_B^2$. Using the results for $\sinh U$ and $\cosh U$ one can write the piece $\hat{H}_1$ of the full Hamiltonian as

$$\hat{H}_1 = \frac{1}{2}[\omega - (E_F^2 - 2E_B^2)] + \omega\gamma^\dagger\gamma \equiv C_1 + \omega\gamma^\dagger\gamma \, , \tag{45}$$

with $C_1 = [\omega - (E_F^2 - 2E_B^2)]/2$. The piece $\hat{H}_2$ has now the form

$$\hat{H}_2 = 2\epsilon_0 E_B(\cosh U - \sinh U)(\gamma^\dagger + \gamma) \equiv C_2(\gamma^\dagger + \gamma) \, , \tag{46}$$

where $C_2 = 2\epsilon_0 E_B(\cosh U - \sinh U)$. The third step requires the diagonalization of $\hat{H}_1 + \hat{H}_2$ in the new form, written in terms of the $\gamma-$operators; this is accomplished by the transformation $\gamma^\dagger = \beta^\dagger + Z$ (with $Z = -C_2/\omega$), leading to

$$\hat{H}_1 + \hat{H}_2 = C_1 - \frac{C_2^2}{\omega} + \omega\beta^\dagger\beta \, , \tag{47}$$

which has the desired diagonalized form. The eigenenergies of the Hamiltonian (47) have the form,

$$\omega_n = C_1 - \frac{C_2^2}{\omega} + \omega n = \frac{1}{2}[\omega - (E_F^2 - 2E_B^2)] - \frac{4\epsilon_0^2 E_B^2 E_F^2}{\omega^2} + \omega n \, , \tag{48}$$

and the ground state obeys the relation

$$\beta|0;\beta\rangle = 0 \Leftrightarrow \gamma|0;\beta\rangle = Z|0;\beta\rangle. \tag{49}$$

One should note that the state $|0;\beta\rangle$ is an eigenstate the γ operator with eigenvalue Z; it is therefore said that $|0;\beta\rangle$ is a coherent state of the operator γ. This last result allows us to write the vacuum of the β operators in terms of the vacuum of the γ operators as

$$|0;\beta\rangle = e^{Z\gamma^\dagger}|0;\gamma\rangle \, , \tag{50}$$

and any eigenstate is written in terms of the β-operators as

$$|n;\beta\rangle = \frac{1}{\sqrt{n!}}(\beta^\dagger)^n|0;\beta\rangle = \frac{1}{\sqrt{n!}}\left(\gamma^\dagger - Z\right)^n e^{Z\gamma^\dagger}|0;\gamma\rangle \, . \tag{51}$$

The eigenenergies ϵ_0^2 of our simpler eigenproblem defined in Eq. (20) are obtained from [see Eq. (35)] $\epsilon_0^2 - \lambda_\pm = \omega_n$, leading to

$$\epsilon_0^2(n,\pm) = \frac{\omega^3}{E_F^4}[n + (1 \pm 1)/2] \, . \tag{52}$$

The double degeneracy $\epsilon_0^2(n,+) = \epsilon_0^2(n+1,-)$ for $n \neq 0$ is immediately recognized. From the analysis we have made in Secs. 2.2 and 2.3 it is now obvious that this degeneracy signals the presence of both solutions $\pm\epsilon_0$ in the original problem [Eq. (15)]. Moreover,

as the eigenvectors of the finite electric field problem have to equal Eq. (28) when $\mathcal{E} \to 0$ we arrive at the following solution,

$$\epsilon(n,\pm) = -e\mathcal{E}l_B^2 k \mp \frac{(E_F^2 - 4E_B^2)^{3/4}}{E_F^{1/2}}\sqrt{n+1} \tag{53}$$

with eigenvectors given by

$$|\psi\rangle_\pm = \sqrt{\left|\frac{E_B}{E_F}\right|}\begin{pmatrix} -|n;\beta\rangle\sqrt{C_+} \mp |n+1;\beta\rangle\sqrt{C_-} \\ |n;\beta\rangle/\sqrt{C_+} \pm |n+1;\beta\rangle/\sqrt{C_-} \end{pmatrix}, \tag{54}$$

where $C_\pm$ is defined in Eq. (33). In addition there is a single non-degenerate solution $\epsilon_0^2(0,-) = 0$, which gives $\epsilon = -e\mathcal{E}l_B^2 k$, and has as eigenvector

$$|\psi\rangle_0 = \sqrt{\left|\frac{E_B}{E_F}\right|}\begin{pmatrix} -\sqrt{C_-} \\ 1/\sqrt{C_-} \end{pmatrix}|0;\beta\rangle\,. \tag{55}$$

This concludes our solution. The eigen-values (53) agree with those obtained by Lukose *et al.*[10]. These authors solved the problem directly in the position basis by transforming the original problem, by means of a Lorentz boost transformation, into a case where the electric field is null.

2.5. Physical interpretation

The standard 2D electron gas pierced by a perpendicular magnetic field is known after Landau[11] to have a spectrum given by,

$$\epsilon(n) = \hbar\omega_{\rm C}\Big(n + \frac{1}{2}\Big), \quad n = 0,1,2,\ldots \tag{56}$$

in complete analogy with the quantum harmonic oscillator, where $\omega_{\rm C} = |eB|/m$ is the cyclotron frequency for electrons with mass m. The so-called Landau levels are equally spaced with level separation $\hbar\omega_c$, which increases linearly with B. An in-plane electric field is easily handled by the transformation $a^\dagger = b^\dagger + e\mathcal{E}l_B/(\hbar\omega_c\sqrt{2})$, whose major consequence is a shift of the entire spectrum,

$$\epsilon(n) = -e\mathcal{E}l_B^2 k - \frac{e^2\mathcal{E}^2 l_B^2}{2\hbar\omega_c} + \hbar\omega_{\rm C}\Big(n + \frac{1}{2}\Big), \quad n = 0,1,2,\ldots \tag{57}$$

with no change for the cyclotron frequency.

Landau levels in graphene are completely different from Landau levels in standard 2D electron gas. As mentioned in Sec. 2.3 graphene's spectrum in perpendicular magnetic field is given by[5],

$$\epsilon(n) = \pm\hbar\tilde{\omega}_{\rm C}\sqrt{n}, \quad n = 0,1,2,\ldots \tag{58}$$

where the cyclotron frequency is $\tilde{\omega}_{\rm C} = \sqrt{2}v_F/l_B$. Two major differences become apparent when comparing Eqs. (58) and (56). Firstly, Landau level spacing in graphene is not constant due to the square root in Eq. (58). Secondly, the standard 2D electron gas has $\omega_{\rm C} \propto B$ whereas graphene shows $\tilde{\omega}_{\rm C} \propto \sqrt{B}$. These dissimilarities are a direct consequence of the effective ultra relativistic nature of the quasi-particles in graphene.

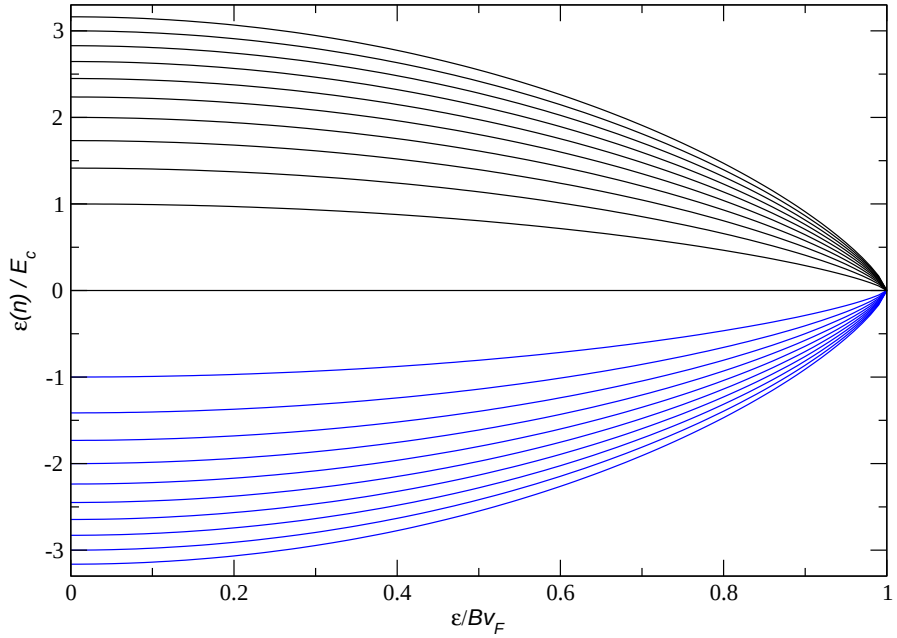

Figure 3. First ten Landau levels as function of $|\mathcal{E}|/v_F|B|$. The quantity E_c is the cyclotron energy $\hbar\tilde{\omega}_c$. The momentum k was chosen to be zero.

As regards the presence of an in-plane electric field and perpendicular magnetic field in graphene we have shown that Landau levels are given by Eq. (53), which can be cast in the form,

$$\epsilon(n) = -e\mathcal{E}l_B^2 k \mp \hbar\Omega_{\rm C}\sqrt{n}, \quad n = 0, 1, 2, \ldots \tag{59}$$

where the new cyclotron frequency reads

$$\Omega_{\rm C} = \sqrt{2}\frac{v_F}{l_B}[1 - \mathcal{E}^2/(B^2 v_F^2)]^{3/4}\,. \tag{60}$$

Thus, unlike the usual 2D electron gas, graphene's cyclotron frequency is renormalized by the electric field, as can be seen in Eq. (60), which, of course, reduces to $\tilde{\omega}_c$ in the limit $\mathcal{E} \to 0$. More important though is the fact that $|\mathcal{E}| \leq v_F|B|$ must be realized if $\Omega_{\rm C}$ is to be real. This last inequality is exactly the same expressed in Eq. (34), and its meaning is now unveiled. As $\mathcal{E}$ approaches $v_F B$ from below $\Omega_{\rm C}$ becomes smaller and smaller, and Landau levels become closer and closer. Eventually, the electric field is such that $|\mathcal{E}| = v_F|B|$, which implies $\Omega_{\rm C} = 0$, and consequent collapse of Landau levels. In Fig. 3 we show the first ten Landau levels (for positive and negative energies) as function of $|\mathcal{E}|/v_F|B|$; the collapse of the Landau levels is clear. For $|\mathcal{E}| > v_F|B|$ the present solution is not valid.

3. Concluding remarks

The problem of a single graphene plane in transverse electric and perpendicular magnetic fields assembles in a simple way several algebraic methods of diagonalizing bilinear

problems. The matrix form of the Dirac Hamiltonian – the low energy effective Hamiltonian for graphene – calls for several non-standard manipulations where canonical transformations and the concept of coherent state appear in a natural way. Furthermore, alike the standard 2D electron gas pierced by a magnetic field, an additional in-plane electric field in graphene induces cyclotron frequency renormalization. Moreover, when the electric field equals the critical value $v_F B$ Landau level collapse is observed.

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